SOLUTIONS OF EXERCISE SHEET 13

Exercise 1. To show that h is well defined, we need to prove that the value of the integral

$$\int_p^z \frac{f'(\xi)}{f(\xi)} d\xi,$$

as defined in the exercise, is independent of our choice of path from p to z over which we integrate. To that end, we remark that since f is non-vanishing on U the function $\frac{f'}{f}$ is holomorphic on U. Furthermore, given that U is simply connected, any closed piecewise C^1 path is contractible. Now, for any $z \in U$ fixed, let γ_1 and γ_2 be two (piecewise) C^1 paths that connect p with z. Then the path

$$\gamma(t) := \begin{cases} \gamma_1(t) \text{ for } t \in [0,1], \\ \gamma_2(s(t)) \text{ for } t \in (1,2] \end{cases}$$

with $s(t) = (2-t)$ is a closed piecewise C^1 path. Therefore,

$$\begin{aligned} 0 = \int_\gamma \frac{f'(\xi)}{f(\xi)} d\xi &= \int_{\gamma_1} \frac{f'(\xi)}{f(\xi)} d\xi + \int_{\gamma_2(s)} \frac{f'(\xi)}{f(\xi)} d\xi \\ &= \int_{\gamma_1} \frac{f'(\xi)}{f(\xi)} d\xi - \int_1^2 \frac{f'(\gamma_2(2-t))}{f(\gamma_2(2-t))} \gamma_2'(2-t) dt \\ &= \int_{\gamma_1} \frac{f'(\xi)}{f(\xi)} d\xi - \int_0^1 \frac{f'(\gamma_2(t))}{f(\gamma_2(t))} \gamma_2'(t) dt \end{aligned}$$

and we see that h is well-defined. To prove that $e^{h(z)} = f(z)$, we compute that

$$\partial_z \left[f(z) e^{-h(z)} \right] = f'(z) e^{-h(z)} - f(z) \frac{f'(z)}{f(z)} e^{-h(z)} = 0$$

So, the function $\frac{f(z)}{e^{h(z)}}$ is constant. Furthermore, by construction, we know that

$$e^{h(p)} = f(p).$$

Thus,

$$\frac{f(z)}{e^{h(z)}} = \frac{f(p)}{e^{h(p)}} = 1 \implies f(z) = e^{h(z)}.$$

Lastly, to obtain a $k-$th root, we simply set

$$g_k(z) = e^{\frac{1}{k} h(z)}.$$

Exercise 2. Consider the function

$$g(z) := f(z) a^{z-1} b^{-z}.$$

Then

$$|g(iy)| = |f(iy)||a^{iy-1}||b^{-iy}| = |f(iy)||a|^{-1} \leq 1$$

and

$$|g(1+iy)| = |f(iy)||a^{iy}||b^{-1-iy}| = |f(1+iy)||b|^{-1} \leq 1$$

for all $y \in \mathbb{R}$. Thus, from exercise 2 of the Monday exercise sheet we infer that

$$|g(z)| \leq 1$$

for all $z \in U$. In other words,

$$1 \geq |f(z)||a^{z-1}||b^{-z}| = |f(z)|a^{\operatorname{Re} z - 1} b^{-\operatorname{Re} z}$$

which immediately implies

$$|f(z)| \leq a^{1-\operatorname{Re} z} b^{\operatorname{Re} z}$$

for all $z \in U$.

Exercise 3. (i) We compute

$$|z-a|^2 = |z|^2 - 2\operatorname{Re}(z\overline{a}) + |a|^2$$

as well as

$$|1 - z\overline{a}|^2 = 1 - 2\operatorname{Re}(z\overline{a}) + |a|^2|z|^2.$$

Thus,

$$\begin{aligned} |\phi_a(z)|^2 \leq 1 &\iff |z-a|^2 \leq |1-z\overline{a}|^2 \\ &\iff |z|^2 + |a|^2 \leq 1 + |a|^2|z|^2 \\ &\iff 0 \leq -|z|^2 - |a|^2 1 + |a|^2|z|^2 \\ &\iff 0 \leq (1-|z|^2)(1-|a|^2). \end{aligned}$$

Hence, we see that$|\phi_a(z)| \leq 1$ on $\overline{D(0,1)}$ with equality if and only if $|z| = 1$.

(ii) Suppose first that f does not vanish on $D(0,1)$. We claim that this implies that f is constant.
To that end, we assume for contradiction that f is not constant. Then, for all $z \in D(0,1)$ we have that $|f(z)| < 1$ by the maximum principle. Since f is non-vanishing by assumption, $\frac{1}{f}$ is also holomorphic on $D(0,1)$ with

$$|\tfrac{1}{f(z)}| = |f(z)| = 1$$

for all $z \in \partial D(0,1)$. However, since $|f(z)| < 1$ on $D(0,1)$ it follows that $|\frac{1}{f}| > 1$. Hence, $\frac{1}{f}$ must have a local maximum on $D(0,1)$ which implies that $\frac{1}{f}$ is constant. This is of course a contradiction to the assumption that f is not constant.

Thus, if f has no zeros and satisfies $|f(z)| = 1$ on $D(0,1)$, it must be of the form $f = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$.
Next, we claim that f has only finitely many zeros. For this we recall that for any domain U the set of zeros of any holomorphic function $f : U \to \mathbb{C}$, aside from the trivial function $f = 0$, cannot have an accumulation point inside of U. Using this, we can once more argue by contradiction. So, assume that there exists a $f \in C(\overline{D(0,1)})$

with $|f(z)| = 1$ for all $z \in \partial D(0,1)$ such that $f : D(0,1) \to \mathbb{C}$ is holomorphic and has infinitely many zeros in $D(0,1)$. We denote these zeros by $\{z_n\}_{n=1}^{\infty}$. Then, since $\overline{D(0,1)}$ is compact, there exists a convergent subsequence of $\{z_n\}$ which we denote by $\{\xi_n\}_{n=1}^{\infty}$ with limit ζ. Furthermore, as ζ is an accumulation point of the zero set of f, we know that necessarily $\zeta \in \partial D(0,1)$. However, since f is continuous on $\overline{D(0,1)}$ this yields

$$1 = |f(\zeta)| = 0$$

and we see that f can have at most finitely many zeros.

With this we can finally prove the statement. So, for any f that satisfies the assumption of the exercise, we let $z_1, z_2, \ldots, z_n$ be its zeros counted according to their multiplicity (i.e., if z is a zero of order m, then we count it m times). Consider now the function

$$g(z) := f(z) \left(\prod_{j=1}^{n} \frac{z - a_j}{1 - z\overline{a_j}} \right)^{-1}$$

By construction, g is non-vanishing on $D(0,1)$ and satisfies $|g(z)| = 1$ on $\partial D(0,1)$. So, by the above consideration g is constant, i.e., $g(z) = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$ and the claim follows.